《基于有限差分的部分饱和双重孔隙介质弹性波模拟与分析》*的补充材料

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Santos 两流一固三相介质模型解析解

考虑一个体力源按照组分的体积含量分配在固体相和流体相上时，对文中（3）式和（8）式作傅里叶变换可以得到频域的 Santos 波动方程：

$$\omega^2\left[\tilde{D}_{11}\tilde{\boldsymbol{u}}+\tilde{D}_{12}\tilde{\boldsymbol{U}}^{(1)}+\tilde{D}_{13}\tilde{\boldsymbol{U}}^{(2)}\right]+N\nabla^2\tilde{\boldsymbol{u}}+\nabla\left[(A+N)\nabla\cdot\tilde{\boldsymbol{u}}+Q_1\nabla\cdot\tilde{\boldsymbol{U}}^{(1)}+Q_2\nabla\cdot\tilde{\boldsymbol{U}}^{(2)}\right]=(1-\phi)\,\tilde{\boldsymbol{f}},$$

$$\omega^2\left[\tilde{D}_{12}\tilde{\boldsymbol{u}}+\tilde{D}_{22}\tilde{\boldsymbol{U}}^{(1)}+\tilde{D}_{23}\tilde{\boldsymbol{U}}^{(2)}\right]+\nabla\left[Q_1\nabla\cdot\tilde{\boldsymbol{u}}+R_1\nabla\cdot\tilde{\boldsymbol{U}}^{(1)}+R_3\nabla\cdot\tilde{\boldsymbol{U}}^{(2)}\right]=S_1\phi\tilde{\boldsymbol{f}},$$

$$\omega^2\left[\tilde{D}_{13}\tilde{\boldsymbol{u}}+\tilde{D}_{23}\tilde{\boldsymbol{U}}^{(1)}+\tilde{D}_{33}\tilde{\boldsymbol{U}}^{(2)}\right]+\nabla\left[Q_2\nabla\cdot\tilde{\boldsymbol{u}}+R_3\nabla\cdot\tilde{\boldsymbol{U}}^{(1)}+R_2\nabla\cdot\tilde{\boldsymbol{U}}^{(2)}\right]=S_2\phi\tilde{\boldsymbol{f}},$$

(A1)

式中，$\tilde{\boldsymbol{u}}$，$\tilde{\boldsymbol{U}}^{(1)}$，$\tilde{\boldsymbol{U}}^{(2)}$ 和 $\tilde{\boldsymbol{f}}$ 分别表示固体相、非润湿相、润湿相和体力的傅里叶变换．引入势函数 φ_1，φ_2，φ_3，$\boldsymbol{\psi}_1$，$\boldsymbol{\psi}_2$，$\boldsymbol{\psi}_3$，Φ，$\boldsymbol{\psi}$，则：

$$\begin{aligned}\tilde{\boldsymbol{u}}&=\nabla\varphi_1+\nabla\times\boldsymbol{\psi}_1,\\ \tilde{\boldsymbol{U}}^{(1)}&=\nabla\varphi_2+\nabla\times\boldsymbol{\psi}_2,\\ \tilde{\boldsymbol{U}}^{(2)}&=\nabla\varphi_3+\nabla\times\boldsymbol{\psi}_3,\\ \tilde{\boldsymbol{f}}&=\nabla\Phi+\nabla\times\boldsymbol{\Psi}.\end{aligned}\qquad\text{(A2)}$$

这里，$\nabla\times\boldsymbol{\psi}_1=\nabla\times\boldsymbol{\psi}_2=\nabla\times\boldsymbol{\psi}_3=\nabla\times\boldsymbol{\psi}=0$．假设声源为胀缩源且 $\Phi=\delta(r)s(\omega)$，将（A2）式代入（A1）式得到非齐次线型方程组：

$$\begin{aligned}&\omega^2\left[\tilde{D}_{11}\varphi_1+\tilde{D}_{12}\varphi_2+\tilde{D}_{13}\varphi_3\right]+\left[P\nabla^2\varphi_1+Q_1\nabla^2\varphi_2+Q_2\nabla^2\varphi_3\right]=(1-\phi)\delta(r)s(\omega),\\&\omega^2\left[\tilde{D}_{12}\varphi_1+\tilde{D}_{22}\varphi_2+\tilde{D}_{23}\varphi_3\right]+\left[Q_1\nabla^2\varphi_1+R_1\nabla^2\varphi_2+R_3\nabla^2\varphi_3\right]=S_1\phi\delta(r)s(\omega),\\&\omega^2\left[\tilde{D}_{13}\varphi_1+\tilde{D}_{23}\varphi_2+\tilde{D}_{33}\varphi_3\right]+\left[Q_2\nabla^2\varphi_1+R_3\nabla^2\varphi_2+R_2\nabla^2\varphi_3\right]=S_2\phi\delta(r)s(\omega),\end{aligned}\tag{A3}$$

（A3）式可以等效转换为齐次线性方程组与声源处正则化条件描述的问题. 为了得到正则化条件，对（A3）式在一个小的球形区域进行体积分，当球的半径趋于零时，根据高斯定理和球形对称条件，有

$$\begin{aligned}&\lim_{\sigma\to 0}\int_{S_\sigma}\left(P\frac{\partial\varphi_1}{\partial r}+Q_1\frac{\partial\varphi_2}{\partial r}+Q_2\frac{\partial\varphi_3}{\partial r}\right)\mathrm{d}s=(1-\phi)\delta(r)s(\omega),\\&\lim_{\sigma\to 0}\int_{S_\sigma}\left(Q_1\frac{\partial\varphi_1}{\partial r}+R_1\frac{\partial\varphi_2}{\partial r}+R_3\frac{\partial\varphi_3}{\partial r}\right)\mathrm{d}s=S_1\phi\delta(r)s(\omega),\\&\lim_{\sigma\to 0}\int_{S_\sigma}\left(Q_2\frac{\partial\varphi_1}{\partial r}+R_3\frac{\partial\varphi_2}{\partial r}+R_2\frac{\partial\varphi_3}{\partial r}\right)\mathrm{d}s=S_2\phi\delta(r)s(\omega).\end{aligned}\tag{A4}$$

根据（A3）式和（A4）式可求得：

$$\begin{aligned}&\varphi_1(r,\ \omega)=\frac{s(\omega)\left(\alpha\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}1}r}+\beta\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}2}r}+\gamma\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}3}r}\right)}{4\pi r},\\&\varphi_2(r,\ \omega)=\frac{s(\omega)\left(\alpha A_{\mathrm{p}_1}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}1}r}+\beta A_{\mathrm{p}_2}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}2}r}+\gamma A_{\mathrm{p}_3}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}3}r}\right)}{4\pi r},\\&\varphi_3(r,\ \omega)=\frac{s(\omega)\left(\alpha B_{\mathrm{p}_1}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}1}r}+\beta B_{\mathrm{p}_2}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}2}r}+\gamma B_{\mathrm{p}_3}\mathrm{e}^{-\mathrm{i}k_{\mathrm{p}3}r}\right)}{4\pi r},\end{aligned}\tag{A5}$$

式中，$k_{\mathrm{p}i}$ 为第 i 种纵波的波数. $A_{\mathrm{p}i}$，$B_{\mathrm{p}i}$，α，β，γ 的表达式满足以下关系式：

$$A_{\mathrm{p}i}=-\frac{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{11}-P & V_{\mathrm{p}_i}^2D_{13}-Q_2\\ V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{23}-R_3\end{vmatrix}}{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{13}-Q_2\\ V_{\mathrm{p}_i}^2D_{22}-R_1 & V_{\mathrm{p}_i}^2D_{23}-R_3\end{vmatrix}},\quad B_{\mathrm{p}i}=-\frac{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{11}-P & V_{\mathrm{p}_i}^2D_{12}-Q_1\\ V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{22}-R_1\end{vmatrix}}{\begin{vmatrix}V_{\mathrm{p}_i}^2D_{12}-Q_1 & V_{\mathrm{p}_i}^2D_{13}-Q_2\\ V_{\mathrm{p}_i}^2D_{22}-R_1 & V_{\mathrm{p}_i}^2D_{23}-R_3\end{vmatrix}},$$

$$\begin{bmatrix} P+Q_1A_{p_1}+Q_2B_{p_1} & P+Q_1A_{p_2}+Q_2B_{p_2} & P+Q_1A_{p_3}+Q_2B_{p_3} \\ Q_1+R_1A_{p_1}+R_3B_{p_1} & Q_1+R_1A_{p_2}+R_3B_{p_2} & Q_1+R_1A_{p_3}+R_3B_{p_3} \\ Q_2+R_3A_{p_1}+R_2B_{p_1} & Q_2+R_3A_{p_2}+R_2B_{p_2} & Q_2+R_3A_{p_3}+R_2B_{p_3} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}=\begin{bmatrix} -s(\omega)(1-\phi)/4\pi \\ -s(\omega)S_1\phi/4\pi \\ -s(\omega)S_2\phi/4\pi \end{bmatrix},$$

当$\eta_{\rm f}^{(1)}=\eta_{\rm f}^{(2)}=0$时，对（A5）式进行反傅里叶变换可求得时域点源激发声场的解析式：

$$\begin{aligned} \varphi_1(r,t)&=\frac{\alpha s(t-r/V_{p_1})+\beta s(t-r/V_{p_2})+\gamma s(t-r/V_{p_3})}{4\pi r}, \\ \varphi_2(r,t)&=\frac{\alpha A_{p_1}s(t-r/V_{p_1})+\beta A_{p_2}s(t-r/V_{p_2})+\gamma A_{p_3}s(t-r/V_{p_3})}{4\pi r}, \\ \varphi_3(r,t)&=\frac{\alpha B_{p_1}s(t-r/V_{p_1})+\beta B_{p_2}s(t-r/V_{p_2})+\gamma B_{p_3}s(t-r/V_{p_3})}{4\pi r}. \end{aligned} \tag{A6}$$

对于沿 y 轴的胀缩线源激发声场解析解，可通过对（A6）式沿 y 轴积分求得

$$\begin{aligned} \varphi_1(x,z,t)&=\frac{\alpha H(t-d/V_{p_1})}{2\pi}\int_{d/V_{p_1}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_1}^2}}{\rm d}\tau \\ &+\frac{\beta H(t-d/V_{p_2})}{2\pi}\int_{d/V_{p_2}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_2}^2}}{\rm d}\tau+\frac{\gamma H(t-d/V_{p_3})}{2\pi}\int_{d/V_{p_3}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_3}^2}}{\rm d}\tau, \\ \varphi_2(x,z,t)&=\frac{\alpha A_{p_1}H(t-d/V_{p_1})}{2\pi}\int_{d/V_{p_1}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_1}^2}}{\rm d}\tau \\ &+\frac{\beta A_{p_2}H(t-d/V_{p_2})}{2\pi}\int_{d/V_{p_2}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_2}^2}}{\rm d}\tau+\frac{\gamma A_{p_3}H(t-d/V_{p_3})}{2\pi}\int_{d/V_{p_3}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_3}^2}}{\rm d}\tau, \\ \varphi_3(x,z,t)&=\frac{\alpha B_{p_1}H(t-d/V_{p_1})}{2\pi}\int_{d/V_{p_1}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_1}^2}}{\rm d}\tau \\ &+\frac{\beta B_{p_2}H(t-d/V_{p_2})}{2\pi}\int_{d/V_{p_2}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_2}^2}}{\rm d}\tau+\frac{\gamma B_{p_3}H(t-d/V_{p_3})}{2\pi}\int_{d/V_{p_3}}^{t}\frac{s(t-\tau)}{\sqrt{\tau^2-d^2/V_{p_3}^2}}{\rm d}\tau. \end{aligned} \tag{A7}$$

